Complete Graph Drawings up to Triangle Mutations

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Abstract. The logical structure we introduce here to describe a (topological) graph drawing, called subsketch, is intermediate between the map (determining the drawing when it is planar), and the sketch introduced by Courcelle (determining the drawing in general but assuming we know the order of the crossings on each edge). For a complete graph drawing, the subsketch is determined, through first order logic formulas, by the size, a corner of the drawing and the crossings of the edges.
We prove, constructively, that two complete graph drawings have the same subsketch if and only if they can be transformed into each other by a sequence of triangle mutations - or triangle switches. This construction generalizes Ringel's theorem on uniform pseudoline arrangements. Moreover, it applies to plane projections of spatial graphs encoded by rank 4 uniform oriented matroids.

Keywords. Graph drawing, logical structure, triangle switch, mutation, pseudoline arrangement, oriented matroid, spatial graph visualization.

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1 Introduction

Three subjects meet in this paper: first the dynamical structure of geometrical objects with triangle mutations (or triangle switches), secondly axiomatics of graph drawings using logical structures as concise as possible, and thirdly the combinatorial study of visualization of spatial graphs encoded by oriented matroids.

In the whole paper, graph drawing is understood in the sense of topological graph drawing, that is drawing of which edges are represented by Jordan arcs (not supposed to be straight), whereas a graph drawing is called geometrical when its edges are represented by (straight) line segments. We consider graph drawings of a graph on a plane where two edges cross at most once and where the unbounded region is defined by the choice of two given adjacent edges called a *corner* (equivalently, we could consider drawings on a sphere, but we would have then to choose a particular point "infinity" so that the region containing it would be considered as the "unbounded" one).

From an axiomatic point of view, a general setting is introduced by Courcelle in [2], allowing both logical and geometrical points of view on graph drawings, and leading to applications of monadic second order logic to graph drawings. In this setting, a graph drawing is determined by its sketch, that is: its underlying graph, the circular ordering of the edges at each vertex, the pairs of edges that cross, and the order of crossings on each edge. If the last data is removed, we get the subsketch of the graph drawing. Hence the subsketch is intermediate between the sketch and the so-called map of the drawing (which determines the drawing if it is planar, see for instance [5]). We prove in Section 3 that, for a complete graph drawing, the subsketch and other useful information, are determined through first order logic formulas by its number of vertices, a corner, and the pairs of edges that cross.

A triangle mutation - or triangle switch - in a graph drawing is passing an edge over the crossing of two other edges, when no obstruction occurs. This local transformation is shown on Figure 1. Obviously a triangle mutation does not change the subsketch. We consider the problem of finding a logical structure for graphs drawings defined up to a sequence of triangle mutations.

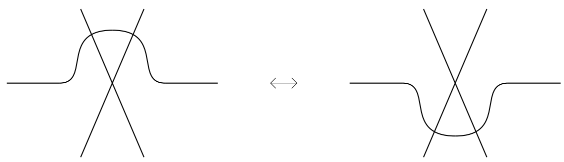

Fig. 1. Triangle mutation (or triangle switch)

We prove constructively in Section 4 that, for a complete graph drawing, the subsketch structure plays this part: it determines the drawing, up to a sequence of triangle mutations and orientation preserving homeomorphisms.

Note that, if one considers a complete graph drawing with an even number of vertices, all of them being drawn on the same circle, then the pairs of opposite vertices define a pseudoline arrangement in a neighbourhood of the centre of the circle, see Figure 2. In fact, the above result generalizes Ringel's theorem on uniform pseudoline arrangements [7] (see Section 5.1).

A consequence of the above result - the original purpose of this paper - is that two projections of complete spatial graphs, defined by finite sets of points in general position representing the same rank 4 uniform oriented matroid [1], are equivalent up to homeomorphism and a sequence of mutations. Hence the combinatorial structure of the oriented matroid together with the logical structure of the projected drawing form the two levels of a modelization of perspectives in spatial graph visualization (see Section 5.2).

In a graph which is not complete, the subsketch is no more sufficient to determine the drawing up to triangle mutations. In general, additional data would be necessary. In this paper, it is an open question. As an example Figure 3 below represents two graph drawings with same crossings and same circular orderings around each edge, but which cannot be tranformed into each other with triangle mutations, since they simply have no triangle.

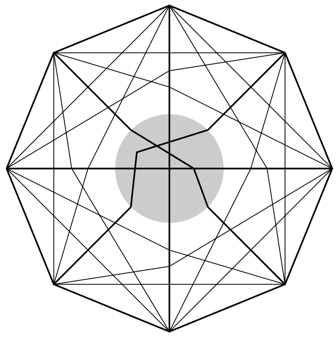

Fig. 2. Complete graph drawing and pseudoline arrangement

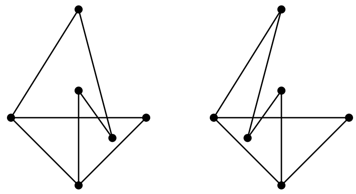

Fig. 3. Two 2-connected graph drawings with same subsketch but no triangle

NB: all proofs of this paper have been removed or shortened in order to fit the requested size for papers of this WG05 proceedings. A full version is forthcoming.

2 Preliminaries

In this paper, a *graph* is always a finite, directed, loop-free, connected graph. The set of vertices of a graph G is denoted V_G, or simply V, and its set of edges is denoted $\overrightarrow{E}_G$, or simply $\overrightarrow{E}$. The underlying undirected set of edges is denoted E_G, or simply E. In fact, the direction of an edge will be used only to define an order of the points on a geometrical representation of this edge. So, for $a, b \in V_G$ and $(a, b) \in \overrightarrow{E}_G$, we will denote $[a, b] = [b, a] \in E_G$.

A *(topological) drawing* of a graph G in the real oriented affine plane is a set of points representing V_G together with a set of drawn edges representing E_G satisfying the following properties:

D1 - a drawn edge is a Jordan arc (i. e. homeomorphic to a closed segment) between the two extremities representing the vertices ; a drawn edge contains no other representation of a vertex of the graph than its extremities.

D2 - two edges having extremities in common (two in the case of multiple edges) meet only at these extremities ; when two edges with no common extremity meet, they cross at this intersection point ; two edges with no common extremity cross at most once.

D3 - no three edges meet at the same point, except if this point is an extremity of the three edges.

Note that if Jordan arcs were replaced by line segments in axiom D1, we would define *geometrical* graph drawings, for which various properties would become trivial (for instance the two Lemmas in Section 3).

With a drawing D of the graph G various pieces of information are associated, encoding the drawing at different levels of abstraction. We call *drawn* element the topological representation of this element in the given drawing.

First the relation $inc_G \subseteq \overrightarrow{E}_G \times V_G \times V_G$ is defined by $(e, x, y) \in inc_G$ if and only if the edge e is directed from the vertex x to the vertex y. Then inc_G describes the structure of the graph G.

Secondly the relation $sig_D \subseteq V_G \times E_G \times E_G$ is defined by $(x, e, f) \in sig_D$ if and only if x is an extremity of e and f, and f is the next edge in the circular ordering around x in the trigonometric sense of rotation, which is well defined by definition of a drawing (property D2).

A *corner* of D is an element $(P, \beta, \alpha) \in sig_D$ such that the drawn vertex P is in the topological boundary of the infinite region of the plane delimited by D, and the intersections of the drawn edges β and α with this boundary are homeomorphic to line segments (containing P). Note that if the graph is complete then β and α are entirely contained in this boundary.

The set of relations inc_G, sig_D define the *map* associated with the drawing D of the graph G. It is well known (see for example [5]) that if D is a drawing with no edge crossing (except for common extremities), and thus G planar, then D is determined up to an orientation preserving homeomorphism of the plane by its map and a corner.

Thirdly, in [2], the relation $dcross_D \subseteq \overrightarrow{E}_G \times \overrightarrow{E}_G$ is defined by $(e, f) \in dcross_D$ if and only if the drawn edges e and f have no extremity in common, the drawn edges e and f have one intersection point and f goes from the left of e to its right when e is directed from bottom to top. Of course $(e, f) \in dcross_D$ implies $(f, e) \notin dcross_D$. In this paper we do not need directed edges for the crossing relation, it is sufficient to consider the relation $cross_D \subseteq E_G \times E_G$, defined by $(e, f) \in cross_D$ if and only if the drawn edges e and f have no extremity in common and the drawn edges e and f have one intersection point. Of course $(e, f) \in cross_D$ implies $(f, e) \in cross_D$. Then we say that $e \in E_G$ and $f \in E_G$ *cross* in D.

The set of relations $inc_G, sig_D, cross_D$ define the *subsketch* of the drawing D.

Fourthly, in [2], the relation $before_D \subseteq \overrightarrow{E}_G \times E_G \times E_G$ is defined by $(e, f, g) \in before_D$ if and only if $f \neq g$, e and f cross in D, e and g cross in D, and the intersection point of e and f is before the intersection point of e and g on the directed drawn edge e. Note that if e crosses f and g then either $before_D(e, f, g)$ or $before_D(e, g, f)$ but not both. The set of relations $inc_G, sig_D, dcross_D, before_D$ define the *sketch* associated with the drawing D, as introduced in [2]. By definition of a drawing, the relation $before_D$ induces, for any edge e, a linear ordering on the elements that cross e. A result of [2] is that the drawing D is determined up to an orientation preserving homeomorphism of the plane by its sketch and its corner.

In view of this result, we will assume from now on that drawings are always given with a certain corner, and are considered up to orientation preserving homeomorphisms (that is an homeomorphism of the plane which preserves the orientation of one - or equivalently any - triangle of the plane). Then *we can identify drawings and sketches*, and the following definitions about drawings or sketches can be made equivalently for one of these two objects, depending on the point of view: geometrical, or logical. When the context is not ambiguous, we may omit the suffix $_D$ referring to the drawing.

Let D be a drawing of a graph G. We call *triangle* of D an element $(e, f, g) \in E_G \times E_G \times E_G$ such that e and f cross in D, e and g cross in D, and f and g cross in D. The order of the elements in the triplet have no importance, and we denote the triangle $[e, f, g]$.

The *segments* of a triangle $[e, f, g]$ are the subsets of the drawn elements e, f, or g which are delimited by the intersection with the two other elements of the triangle. The *interior* of a triangle $[e, f, g]$ is the bounded region of the plane delimited by its segments and containing these segments. A triangle is *contained* in another triangle if the two triangles are not equal, they have two common elements, and the interior of the first one is contained in the interior of the second one. We say that $h \in E_G$ *cuts the triangle* $[e, f, g]$, resp. *cuts the triangle* $[e, f, g]$ *twice*, if, geometrically, the drawn element h has a non empty intersection with at least one, resp. two, segment(s) of $[e, f, g]$. The following easy Lemma 1 is illustrated by Figure 4.

Lemma 1. *If* $[i, j, k]$ *is a triangle cut twice by* e*, then one and only one triplet in* $\{\ \{i, j, e\},\ \{i, k, e\},\ \{j, k, e\}\ \}$ *defines a triangle contained in* $[i, j, k]$. □

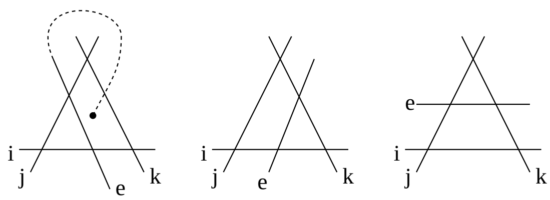

Fig. 4. Triangle $[i, j, k]$ cut twice by e

Let D and D' be two drawings of the graph G with same subsketch. As D and D' have the same *cross* relation, they have same triangles. We say that a triangle $[e, f, g]$ is *permuted between* D *and* D' if the ordering of crossings between its edges along each of its three edges is different in the two drawings, that is if $before_D(e, f, g) = \neg before_{D'}(e, f, g)$, $before_D(f, e, g) = \neg before_{D'}(f, e, g)$, and $before_D(g, e, f) = \neg before_{D'}(g, e, f)$, We call *free* a triangle of which interior has an intersection with the drawing reduced to the segments of the triangle. In particular it is not cut by any element, but not that the converse is false as show the triangle $[e, k, i]$ in the left Figure 4 when j is removed.

Given a drawing D of a graph G and a free triangle $[e, f, g]$ of D, the *mutation of* $[e, f, g]$ *from* D is the sketch D' of G for which all relations are the same as in D, except that e and f, and resp. e and g, and resp. f and g, are permuted on the drawn edge g, and resp. f, and resp. e. In other words all relations are the same in D' as in D except that the triangle $[e, f, g]$ is permuted between D and D'. We denote $D \to D'$, and call $[e, f, g]$ the *mutated* triangle from D to D'.

Hence, a triangle $[e, f, g]$, which is free in D, is permuted between D and its mutation from D. But, of course, a triangle may be permuted between two drawings D and D', without being free in D nor in D'.

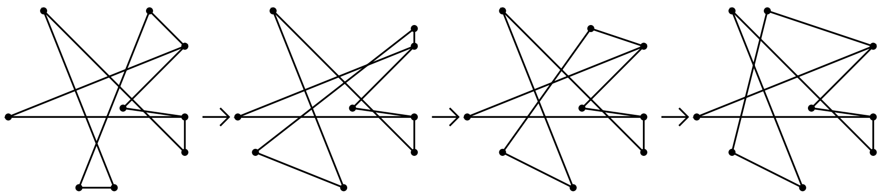

Fig. 5. A sequence of mutations

A *sequence of mutations* from the sketch of a drawing D is a sequence of sketches, each one being the mutation of a free triangle from the previous one. On the example of Figure 5, the triangle containing a vertex cannot be mutated, but the three other triangles can be mutated triangles in a sequence of mutations.

3 Logical structure of complete graph drawings

In this section, we prove that, for a complete graph drawing with given number of vertices and given corner, the *cross* relation is sufficient to determine, through first order logic formulas, not only the the *sig* relation and thus the subsketch of the drawing, but also an *ins* relation which states if a vertex of a graph is inside the triangle formed by three other vertices. This is not true for general graph drawings (see Figure 3). We shall see that these relations determine also several other relations and finally determine the sketch of the drawing except the $before$ relations for edges of triangles containing no vertex.

Let D be a graph drawing, with corner (P, β, α). The vertex P is called *vertex at the corner*, and the other extremities of α and β are denoted respectively A and B.

For three vertices $e, f, g \in V_G$, we denote $[e, f, g]$ the bounded region of the plane delimited by the drawn edges $[e, f]$, $[f, g]$ and $[g, e]$, containing these drawn edges. Thus this region does not contain the vertex at the corner P when $P \notin \{e, f, g\}$. Not that by definition, such a region is equivalent to a closed ball up to homeomorphism. The relation $ins_D \subseteq V_G \times V_G \times V_G \times V_G$ is defined by $(x, e, f, g) \in ins_D$ if and only if $x \notin \{e, f, g\}$ and the drawn vertex x is inside the region $[e, f, g]$.

For the construction of the next theorem, we introduce a relation $bet_D \subseteq V_G \times E_G \times E_G \times E_G$ called *between* relation for the drawing D, such that $(x, e, f, g) \in bet_D$ if the edges e, f, g all have extremity x, and f is between e and g in the circular order of the edges around x (note that the order is essential in the sentence: f is not between g and e).

The *size* of a complete graph drawing is the number of vertices of the underlying complete graph.

Theorem 1. *The subsketch and the inside relation of a complete graph drawing are determined, through first order logic formulas, by its size, its crossing relation and its corner.*

Proof. The construction is step by step and uses extensively the topological definition of the corner and properties (D1) (D2) (D3) of a drawing.The proof is not difficult and is about two pages long. However the ordering of the steps is important. Briefly: begin with the inside relations for triplets containing P, then for general triplets, then consider the between relations around P, and then the between relations around any vertex. □

Since the *sig* relations are determined, we easily get the following corollary by using the restrictions to 4 vertices subdrawings.

Corollary 1. *Let D be a complete graph drawing. Its dcross relation is determined with first order logic formulas by its size, crossing relation and corner.* □

The following results are trivial in the geometrical case. They generalize to topological graph drawings, quite technically but easily, using Theorem 1 and the axioms (D1), (D2), (D3), by considering the several possible representations.

Lemma 2. *Let D be a complete graph drawing with given size, crossing relation and corner. Let f and g be two edges such that either f and g have same extremity, or f and g do not cross. If f and g both cross an edge e, then the $before(e, f, g)$ relation is determined by first order logic formulas.* □

Corollary 2. *Let D and D' be two complete graph drawings with same size, crossing relation and corner. Then $D \neq D'$ if and only if there exists a permuted triangle between D and D'.* □

We say that a drawn triangle T *contains* a drawn vertex a, if the drawn vertex a is inside the bounded region of the plane delimited by drawn edges of T

Lemma 3. *Let D be a complete graph drawing, with given size, crossing relation and corner. Let $T = [e, f, g]$ be a triangle, and a a vertex of D. The property that the drawn triangle T contains the drawn vertex a is expressible by a first order logic formula. Moreover, when this property is true for some a, the $before(e, f, g)$ relation is also determined by a first order logic formula.* □

Corollary 3. *If two complete graph drawings have same size, crossing relation and corner, then a drawn triangle permuted between the two sketches contains no drawn vertex of the graph.* □

4 Triangle mutations in complete graph drawings

In the previous Section we saw that two complete graph drawings with same corner and subsketch have the same $before$ relations except for triangles containing no drawn vertex. The aim of this Section is to prove that two complete graph drawings with same corner have same subsketch if and only if they can be transformed into each other by a sequence of mutations. The "if" way is obvious since a mutation does not change the subsketch, the "only if" way is made by an algorithm.

For a drawing D of a graph G, and a drawn edge e of D, we denote $D - e$ the drawing obtained by removing the drawn edge e except the intersection points with other edges. Note that if $G - e$ is not connected, then an extremity a of e is isolated in $G - e$, and by definition is not represented in $D - e$.

Let G be a complete graph with vertices $\{a_1, ..., a_n\}$, the (undirected) edges of G are denoted $e_{i,j} = [a_i, a_j]$, $1 \leq i < j \leq n$. For a drawing D of G, we denote $D_n = D$ and, for $1 \leq i < n$, $D_i = D - \{e_{i,n}, e_{i+1,n}, ..., e_{n-1,n}\}$. In particular, D_1 is a drawing of the complete graph on $n - 1$ vertices $a_1, ..., a_{n-1}$. When D is given with a corner (P, β, α), we choose to numerate vertices so that $P = a_1$, $\beta = [a_1, a_2]$ and $\alpha = [a_1, a_3]$, so that it remains a corner of the considered subdrawings.

Lemma 4. *Let $1 \leq i < n$, and let D and D' be two complete graph drawings, with same size, crossing relation and corner, such that $D_i = D'_i$. Then there exists a permuted triangle between D_{i+1} and D'_{i+1}, and a sequence of mutations from D_{i+1} to D'_{i+1} containing only permuted triangles between D_{i+1} and D'_{i+1}.*

Proof. The proof is about one page long and consists in a sweeping of e_i. □

Theorem 2. *Let D and D' be two complete graph drawings with same size, crossing relation and corner. There exists a sequence $S(D, D')$ of mutations $D = D^{(0)} \rightarrow D^{(1)} \rightarrow ... \rightarrow D^{(k-1)} \rightarrow D^{(k)} = D'$ from D to D'. Moreover this sequence can be chosen such that, for any intermediate sketch $D^{(i)}$, $1 \leq 0 \leq k-1$ the mutated triangle from $D^{(i)}$ to $D^{(i+1)}$ is contained in a permuted triangle between $D^{(i)}$ and D'. It is given by the following algorithm.*

<u>Computation of the first triangle $T(D_i, D'_i)$ from D_i to D'_i</u>

if $n \leq 3$ or $D_i = D'_i$ then $T(D_i, D'_i) = \emptyset$
if $n > 3$ and $1 < i \leq n$ then let $T = T(D_{i-1}, D'_{i-1})$
  if $T \neq \emptyset$ then
    if T is free in D_i then $T(D_i, D'_i) := T$
    otherwise T is cut by $e_{i,n}$ in D_i then there exists (by lemma 1) a unique T'
      contained in T, free in D_i, with $e_{i,n} \in T'$, and $T(D_i, D'_i) := T'$
  if $T = \emptyset$ then there exists (by lemma 4) T', free in D_i, with $e_{i,n} \in T'$,
      permuted between D_i and D'_i, and $T(D_i, D'_i) := T'$ (arbitrary choice)

<u>Computation of $S(D, D')$</u>

if $T(D, D') = \emptyset$ then $S(D, D') := D$
otherwise D'' being obtained by mutation of $T(D, D')$ from D

$$S(D, D') := D \to S(D'', D')$$

Proof (sum up). We prove Theorem 2 by induction on n and $1 < i \leq n$, using the previous algorithms. Recall that D_1 is a drawing of the complete graph on $n - 1$ vertices, hence $T(D_1, D'_1)$ and $S(D_1, D'_1)$ are built for drawings of K_{n-1}. Note that, by Corollary 2, for all $1 < i \leq n$, we have $D_i \neq D'_i$ if and only if there exists a permuted triangle between D_i and D'_i.

The direct computation of $S(D_i, D'_i)$ can be done the following way: first build $S(D_i - e_{i,n}, D_i - e_{i,n}) = S(D_{i-1}, D'_{i-1})$. The key point is that any triangle in this sequence at level $i - 1$ is contained by induction hypothesis in a triangle which is permuted between the current sketch and the final one. Hence it cannot contain a vertex of the graph according to Corollary 3. So free triangles used in the sequence of mutations at level $i - 1$ which are not cut by $e_{i,n}$, remain free triangles at level i.

Then add the mutations built in the algorithm when $T \neq \emptyset$ and T is cut by $e_{i,n}$ using Lemma 1. These added mutations all contain $e_{i,n}$. The sequence obtained here is denoted S'', and the arrangement obtained from D_i by S'' is D''_i. Then $D''_{i-1} = D'_{i-1}$ and by Lemma 4 there exists a sequence S'' from D'_i to D'_i using only mutations containing $e_{i,n}$. Then $S = S' \to S''$ is a sequence of mutations from D_i to D'_i.

At last, $T(D_i, D'_i)$ is contained in a permuted triangle between D_i and D'_i: either $T(D_{i-1}, D'_{i-1}) = \emptyset$ and it is a permuted triangle between D_i and D'_i, or it is contained in $T(D_{i-1}, D'_{i-1})$, which is contained in a permuted triangle between D_{i-1} and D'_{i-1} (by induction hypothesis), and so between D_i and D'_i. □

5 Examples and applications

5.1 Triangle mutations in pseudoline arrangements

A *pseudoline arrangement* may be defined as a finite set of curves in the affine plane, each one being homeomorphic to a line, and such that any two pseudolines cross each other exactly once. We will always consider *uniform* pseudoline arrangements, i. e. no three pseudolines can meet at the same point. We consider that a pseudoline arrangement is labelled and given with the *circular ordering* of the pseudolines at infinity, and is defined up to an orientation preserving homeomorphism. Pseudoline arrangements (equivalent to rank 3 oriented matroids) are well studied objects, see [1] chapter 4. They satisfy simple axiomatics with the $before$ relation [1], and even first order axiomatics [3].

Here, a pseudoline arrangement can be considered as a structure similar to a sketch of which inc and sig relations are not useful, of which crossing relation is trivial (each element crosses each other element once), and determined, when each pseudoline is directed, by the linear ordering of the crossings on each pseudoline, that is by a $before$ relation. Hence all definitions about triangles and mutations can be done exactly the same way in pseudoline arrangements. So the previous result and algorithm apply naturally: for an arrangement A on $E = \{e_1, ..., e_n\}$, we denote A_k, $1 \leq k \leq n$, the arrangement on $E_k = \{e_1, ..., e_k\}$ obtained by restriction from A, and we replace D_i with A_i and $e_{i,n}$ with e_i in Theorem 2. Note that a similar natural inductive construction for a sequence of mutations has been used for pseudoline arrangements by Roudneff in [8].

The well known Ringel's theorem on pseudoline arrangements [7] states that if A and A' are two uniform pseudoline arrangements with same number of elements and same circular ordering at infinity then there exists a sequence of mutations from A to A'. Hence Theorem 2 gives a slight strengthening of this theorem, which allows to transform A into A' avoiding mutations of triangles not contained in a permuted triangle. Indeed, in the generalization to graph drawings, we want to avoid mutations of triangles containing drawn vertices.

The very important point is that it is not possible in general to transform a configuration into another one using only mutations of permuted free triangles, as it would mean there is always a permuted free triangle between two different configurations, which is false as shown on the example below. This has been mentioned in [4] from which Figure 6 is taken and made straight. Note that one of these two arrangements had already been a significant example for another problem in [1] Figure 1.11.2.

Example. The sequences of triangles built by the previous algorithm applied to the arrangements of Figure 6 are the following. We separate the two built subsequences: the first one (S' in the proof of Theorem 2) built from the previous level, and the second one when only the last pseudoline has to be moved (S'' in the proof of Theorem 2).

- at level 3: $\emptyset$ (triangles 123 are the same in both arrangements)

- at level 4: ($\emptyset$) $\rightarrow$ (234 $\rightarrow$ 134 $\rightarrow$ 124) (only 4 has to be moved)

- at level 5: (235 $\rightarrow$ 234 $\rightarrow$ 135 $\rightarrow$ 134 $\rightarrow$ 125 $\rightarrow$ 124) $\rightarrow$ ($\emptyset$) (the first is sufficient)

- at level 6: (**356** $\rightarrow$ 235 $\rightarrow$ 346 $\rightarrow$ 234 $\rightarrow$ 135 $\rightarrow$ 134 $\rightarrow$ 125 $\rightarrow$ 124) $\rightarrow$ (236 $\rightarrow$ 126 $\rightarrow$ 136 $\rightarrow$ 146 $\rightarrow$ 156 $\rightarrow$ 456 $\rightarrow$ 256 $\rightarrow$ **356**)

This example shows two pseudoline arrangements having all their free triangles (123, 145, 356 and 246) in the same position. Then a sequence of mutations

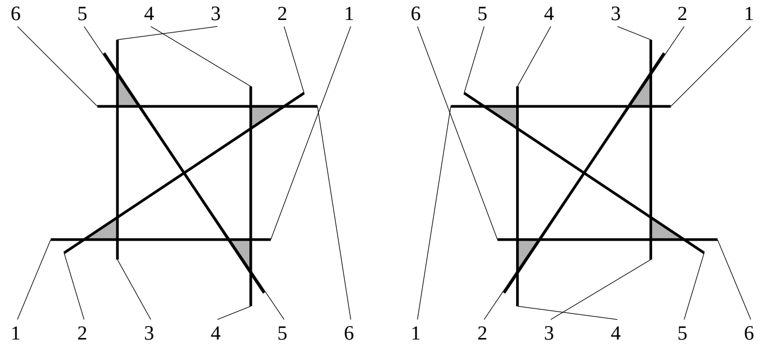

Fig. 6. Two arrangements with no permuted free triangle

from one to the other must begin with the mutation of a non permuted triangle. Hence the minimal number of mutations needed in the sequence may be strictly larger than the number of permuted triangles. For instance in the above sequence, we used twice the mutation of 356. The problem of building a minimal sequence of mutations in general is open.

5.2 Visualization of spatial graphs encoded by oriented matroids.

Consider a set E of $n+1$ points in the 3-dimensional real (or rational) space in general position, a plane in general position with this configuration, and $a \in E$ the extremal point in E with respect to the plane (i. e. the distance from a to the plane is maximal). Then the projections, from a to the plane, of the segments formed by all pairs of vertices is a complete (geometrical) graph drawing on n vertices (see Figure 7).

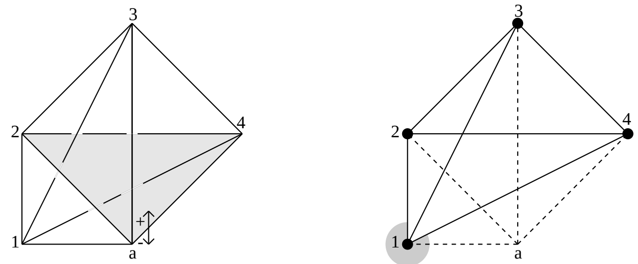

Fig. 7. Perspective on a spatial graph

Theorem 3. *The rank 4 oriented matroid defined by E determines a corner and the cross relations of the drawing obtained by projection from the extremal point $a \in E$. Hence it determines the drawing up to a sequence of triangle mutations.*

Proof. With the oriented matroid, we know for each triplet in E, and for each pair of other points, if these two points are on the same side or the opposite sides of the plane spanned by the triplet, i. e. we know the relative signs of elements in a cocircuit defined by the triplet. Then we easily get a corner of the drawing and its *cross* relations (but not all the drawing). We end using Theorem 2. □

With theorem 3 we know that if two such configurations of points define the same oriented matroid up to a bijection of the ground set, then their projections, from extremal points being in bijection, are the same up to a sequence of triangle mutations and orientation preserving homeomorphisms.

Note that this application uses mainly particular cases of the constructions of the paper because: first, the graph drawing obtained by projection is a geometrical graph drawing, that is a drawing with straight edges, and secondly, the oriented matroid structure may determine directly the inside and map relations on the drawing.

Note nevertheless that the obtained result is not trivial since it is impossible in general to transform the first point configuration into the second by an isotopy of the space preserving the oriented matroid structure (which would have been, if true, an immediate way to build the required sequence of mutations). This fact is known in oriented matroid theory [1] as the *Universality Theorem of Mnëv*, stating that realization spaces of oriented matroid are not connected, and in fact are birationally equivalent to semi-algebraic varieties. For some other spatial transformation problems related to spatial graphs, see [6].

Finally, the point a plays the part of a point of view. When a moves in a region delimited by the planes formed by other points of the configuration, the oriented matroid data, and the subksetch, are unchanged, but the drawing, and its sketch, change with a sequence of triangle mutations. When a crosses a plane, the oriented matroid data changes (a sign changes in some cocircuit). Thus, it is a certain modelization, using two structural levels, of spatial graph visualization.

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